Lecture 1

6.1 - Inverse Functions

The inverse of a function is given by running it backwards. A natural question then is: What conditions must a function, f, satisfy so that we can invert it? (i.e., when can we find f^{-1}?)

We need the following condition:

Def: A function f is one-to-one if it never takes the same value twice, i.e., if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Ex: The functions $f(x) = x$ & $f(x) = x^3$ are one-to-one, but $f(x) = x^2$ & $f(x) = \cos x$ are not.

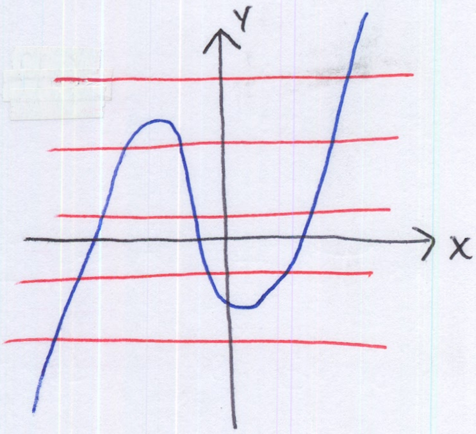
How do we determine whether functions are one-to-one? One way to do this is by using the graph of the function and the Horizontal Line Test.

Horizontal Line Test

A graph passes the horizontal line test if each horizontal line hits the graph at most once.

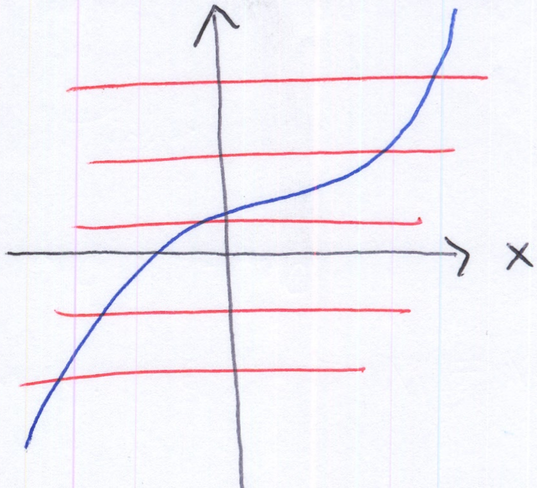
Ex: Use the Horizontal Line Test to determine whether the following functions are invertible:

a)



Invertible? NO

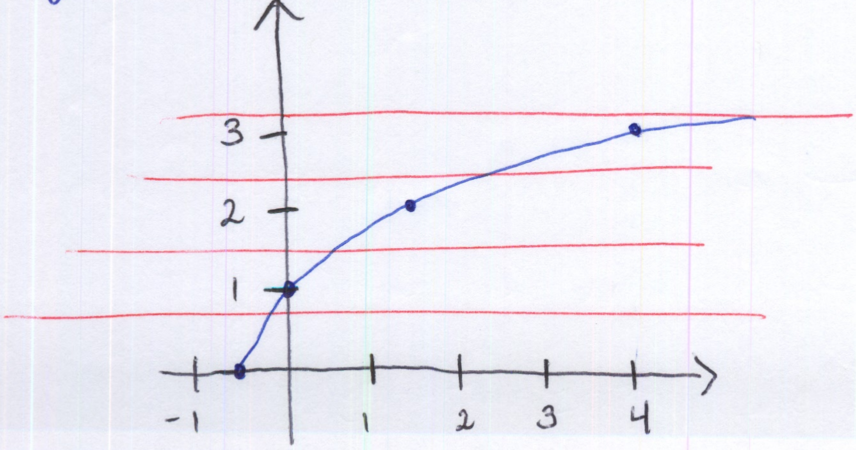
b)



Invertible? YES

Ex: Is $f(x)=\sqrt{2x+1}$ invertible?

The graph of $f(x)=\sqrt{2x+1}$ passes the HLT:



So, yes, it is invertible.

Inverse functions

Given a one-to-one function f with domain A and range B, we define the inverse function f^{-1} via the rule

$$\boxed{f^{-1}(y) = x \quad \text{if and only if} \quad y = f(x)}$$

Ex: Use the above rule to find $f^{-1}(9)$ and $f^{-1}(65)$ where $f(x) = x^3 + 1$.

$f(2) = 8 + 1 = 9$, so $f^{-1}(9) = 2$.

$f(4) = 64 + 1 = 65$, so $f^{-1}(65) = 4$.

Note: domain f = range f^{-1} & range f = domain f^{-1}

I'll use $D(f)$ for the domain of f and $R(f)$ for the range of f.

Ex: Let $g(x) = \sqrt{2x+1}$. Fill in the chart:

Typo in original notes. Change 81 to 9.

Does g^{-1} exist?	$D(g)$	$R(g)$	$g^{-1}(9)$
Yes. It passed the HLT.	$[-\frac{1}{2}, \infty)$	$[0, \infty)$	$g(40) = \sqrt{81} = 9$ $\Rightarrow g^{-1}(9) = 40$

Given a function, f, how do we find its inverse, f^{-1}?

1) Write $y=f(x)$.
2) Switch x & y everywhere
3) Solve for y to get $y=f^{-1}(x)$.

Ex: Let $f(x)=\dfrac{4x-1}{2x+3}$. Find a formula for $f^{-1}(x)$.

$$y=\frac{4x-1}{2x+3} \xleftrightarrow{\text{switch } x \,\&\, y} x=\frac{4y-1}{2y+3} \iff x(2y+3)=4y-1$$

$$\iff 2xy-4y=-3x-1$$

$$\iff y(2x-4)=-3x-1$$

$$\iff y=\boxed{f^{-1}(x)=\frac{-3x-1}{2x-4}}$$

Composing f and f^{-1}

We have if $x=f^{-1}(y)$, then $y=f(x)$. Putting these together, we have $\boxed{x=f^{-1}(f(x))}$

likewise, we have if $x=f(y)$, then $y=f^{-1}(x)$ and

$$\boxed{x=f(f^{-1}(x))}$$

We can verify this with our example above:

$$f(f^{-1}(x)) = \frac{4\left(\frac{-3x-1}{2x-4}\right)-1}{2\left(\frac{-3x-1}{2x-4}\right)+3} = \frac{\frac{-6x-2}{x-2}-1}{\frac{-3x-1}{x-2}+3} = \frac{\frac{-6x-2-x+2}{x-2}}{\frac{-3x-1+3x-6}{x-2}}$$

$$= \frac{\frac{-7x}{x-2}}{\frac{-7}{x-2}} = x$$

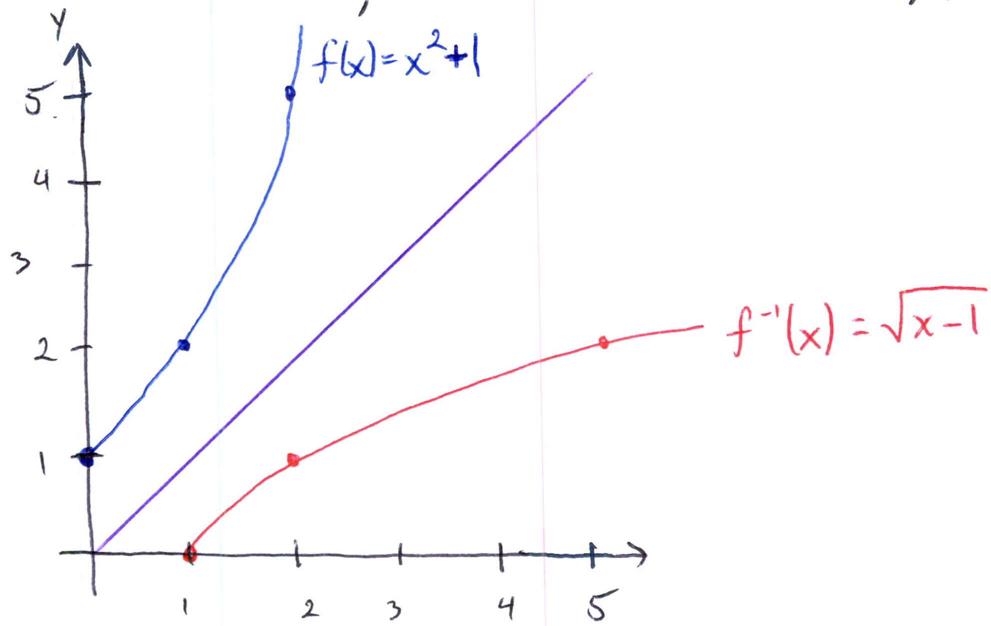
$$f^{-1}(f(x)) = \frac{-3\left(\frac{4x-1}{2x+3}\right)-1}{2\left(\frac{4x-1}{2x+3}\right)-4} = \frac{\frac{-12x+3}{2x+3}-1}{\frac{8x-2}{2x+3}-4} = \frac{\frac{-12x+3-2x-3}{2x+3}}{\frac{8x-2-8x-12}{2x+3}}$$

$$= \frac{\frac{-14x}{2x+3}}{\frac{-14}{2x+3}} = x$$

Sometimes we make up notation for inverse functions (only when doing so is useful) as in e^x & $\ln x$, or $\tan x$ & $\arctan x$. Sometimes though, there is no way to compute the inverse, though the inverse does exist, e.g., for $f(x) = xe^x$. However, if we have the graph of $f(x)$, we can always find the graph of $f^{-1}(x)$.

The graph of $y=f^{-1}(x)$ is given by reflecting the graph of $y=f(x)$ about the line $y=x$.

<u>Ex</u>: Consider $f(x)=x^2+1$, with domain $[0,\infty)$. Graph $f^{-1}(x)$.



Calculus with Inverse Functions

<u>Theorem</u>: If f is one-to-one and continuous on an interval, then its inverse is also one-to-one and continuous on an interval.

Remember that $x=f(f^{-1}(x))$. Using the chain rule, we differentiate both sides to get:

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}\left(f(f^{-1}(x))\right) = f'(f^{-1}(x)) \cdot \left((f^{-1})'(x)\right)$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

<u>Theorem</u>: If f is a one-to-one, differentiable function with inverse function $f^{-1}(x)$, and $f'(f^{-1}(a)) \neq 0$, then f^{-1} is differentiable at a, and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

<u>Ex</u>: Find $(f^{-1})'(65)$ where $f(x) = x^3 + 1$.

We know $f^{-1}(65) = 4$, so we need $f'(4)$.

$f'(x) = 3x^2 \Rightarrow f'(4) = 48$.

$$\Rightarrow (f^{-1})'(65) = \frac{1}{f'(f^{-1}(65))} = \frac{1}{f'(4)} = \frac{1}{48}$$

Notice that it wasn't necessary to find $f^{-1}(x)$, but all we needed was the value $f^{-1}(65)$.

Ex: If f is a one-to-one, differentiable function, and that $f(4)=5$, $f'(4)=\frac{2}{3}$, $f^{-1}(4)=6$, and $f'(6)=\pi$, find $(f^{-1})'(4)$.

$$(f^{-1})'(4)=\frac{1}{f'(f^{-1}(4))}=\frac{1}{f'(6)}=\frac{1}{\pi}$$